

## A PARADOX OF RELEVANT IMPLICATION

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### 1. *Background*

The paradoxes of material implication are based on the observation that any true proposition is materially implied by any other, and that any false proposition materially implies any other. This makes it impossible to believe both that such relations do not hold for implication itself and that material implication is the correct technical account of implication. Of course it cuts two ways: one can infer either that indeed material implication is not implication itself, or that one's intuitions about implication were wrong.

Similarly, the paradoxes of strict implication are based on the observation that any necessarily true proposition is strictly implied by any other, and that any necessarily false proposition strictly implies any other. This makes it impossible to believe both that such relations do not hold for implication, or entailment, itself and that strict implication is the correct technical account of entailment. Again, it cuts two ways: one can infer either that indeed strict implication is not entailment, or that one's intuitions about implication and entailment were wrong.

In reaction to these paradoxes, calculi of relevant implication and of entailment have been set up.<sup>1</sup> I wish to point out a curious feature of one of these systems to which one may react similarly: either the calculus of relevant implication does not correctly capture implication, or one's intuitions about implication are at fault.

### 2. *The paradox*

An argument is sound if it is valid and has true premises. The standard way to prove a result is to find a sound argument of which it is the conclusion.

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<sup>1</sup> A. R. Anderson and N. D. Belnap, *Entailment*, vol. I (Princeton, 1975).

Let us call an argument *supersound* if its validity is merely compatible with the truth of its premises. That is, for such an argument, that it is valid does not imply that (one or more of) its premises are false. Then every sound argument is supersound, but not necessarily conversely. There may be supersound arguments which are both invalid and have (one or more) false premises.

However, the conclusion of every supersound argument is true. For suppose it were false; then if the argument were valid, (at least one of) its premises would need be false too, that is, the validity of the argument would be incompatible with the truth of its premises.

So in order to demonstrate the truth of a claim we do not need to find a sound, or even a valid, argument for it. All we need is an argument which *could* be sound, that is, could both be valid and have true premises.

In consequence, it can no longer constitute a rejection of an argument in support of some claim that the argument is simply invalid, or that it has false premises. For such criticism leaves open the possibility that, though unsound, it is nonetheless supersound.

There are, it seems, invalid arguments with false premises that can nevertheless be used to show that their conclusions are true.<sup>2</sup>

### 3. *Chicken and egg*

Let us call any argument *hypersound* whose conclusion is true. Clearly every sound argument is hypersound, but there may be, indeed there clearly are, unsound hypersound arguments, and trivially the conclusion of every hypersound argument is true. So it would seem that we do not need a sound argument to support a claim, or even a supersound one, but only a hypersound one.

But hypersoundness is a hollow notion: for to show an argument hypersound we have precisely to show what we are using the hypersound argument to prove, namely that its conclusion is true. So to infer from the fact that an argument is hypersound that its conclusion is true is circular.

<sup>2</sup> The ideas of §§ 2 and 4 were presented to the Australasian Association of Logic Conference at Monash University in October 1979; an abstract appears in the *Journal of Symbolic Logic*, 46 (1981), 426, under the title, "Supersound Arguments".

With a sound argument this is not so — there is no circularity. We can often, we want to say, show an argument sound without first demonstrating the truth of its conclusion, that is, independently of showing its conclusion to be true.

This notion of independence needs some clarification. For in one sense we cannot show an argument sound independently of showing its conclusion true, since the truth of the conclusion immediately follows. But what is important here is that there is no step from the truth of the conclusion to the soundness of the argument — not even by the satisfaction of some further condition. Showing the conclusion true is no help in showing an argument sound. What we do is first show the argument valid, then show the premises true. Having shown this, it follows that its conclusion is true.

What of supersound arguments? Can we show an argument to be supersound without first showing either that it is sound or that its conclusion is true? That we do not need to infer supersoundness from soundness will be the topic of §§ 5 and 6. We will find that one can demonstrate the compatibility of certain arguments' validity and the truth of their premises independently of showing their conclusions true. Moreover, that supersoundness is independent of the truth of the conclusion in the appropriate sense, holds just as for soundness. The truth of the conclusion cannot be used to show an argument supersound. Other means are needed to show an argument supersound. It then follows, and not circularly, that the conclusion is true.

### 4. *A modal fallacy*

To proceed, we need to identify entailment and compatibility technically. We consider a general form of argument  $P/Q$ , from premises  $P$  to conclusion  $Q$ . " $P/Q$  is valid" means that  $P$  entails  $Q$ , "the validity of  $P/Q$  is compatible with the truth of  $P$ " means that it is false that if  $P$  entails  $Q$  then  $P$  is false. Note that we must distinguish implication from entailment, for an argument is valid and its premises entail its conclusion just when the conditional with (the conjunction of) the premises as antecedent and the conclusion as consequent is necessarily true, that is, when the premises necessarily imply the conclusion. Implication is, in general, a contingent relation between propositions: as things are, if the premises are true so is the conclusion. One might wonder whether there

are any such contingent truths about implication; how, one might challenge, could one verify a supposed contingent statement of implication? One example of a contingent conditional is the causal conditional. A causal conditional is one whose truth consists in the holding of a causal relation between two events. This relation can be shown to hold, even if it cannot be (conclusively) verified, by adducing evidence of an empirical nature. Even if we must concede (and perhaps we must) that some empirical truths are necessarily true, in this case the empirically established truths are ones that could have been different, and so are (in general) contingent. Of the four accounts of supersoundness we will consider, two ground compatibility on this contingent relation of implication, expressed by a bare conditional; and two ground it on entailment.

(i) If "entails" (validity) were correctly characterised by strict implication and "if" by material implication, all supersound arguments would be sound. For then the supersoundness of  $P/Q$  would require the truth of  $\sim((P \rightarrow Q) \supset \sim P)$ , that is, of  $(P \rightarrow Q) \& P$ , and this is just to demand soundness of  $P/Q$ . Hence this classical position does not distinguish compatibility from conjunction, or joint truth, and so cannot distinguish supersoundness from soundness.

(ii) If "entails" is " $\rightarrow$ " and so is the "if" of compatibility, then the argument for the truth of the conclusion of every supersound argument commits a modal fallacy. For the argument runs:

Clearly

$$(P \rightarrow Q) \rightarrow (\sim Q \rightarrow \sim P)$$

is true, by contraposition. So by Permutation

$$\sim Q \rightarrow ((P \rightarrow Q) \rightarrow \sim P)$$

is true, that is, the truth of  $\sim Q$  strictly implies (i.e. entails) that  $P/Q$  is not supersound. Hence again by contraposition, that  $P/Q$  is supersound entails that  $Q$  is true.

But although the other steps are valid, Permutation is not a permissible move with a modal connective such as " $\rightarrow$ ".  $B \rightarrow (A \rightarrow C)$  follows from  $A \rightarrow (B \rightarrow C)$  only when  $B$  is a necessitive, that is, when  $B$  is

equivalent to  $\Box D$  for some  $D$ . We can conclude only that the conclusion of every supersound argument is possibly true, and that every supersound argument with a necessitive as conclusion has indeed a true conclusion (in the same way that the material implicational paradoxes reappear for strict implication restricted to necessitives). The upshot is that this second classical position indeed avoids both the identification of compatibility with joint truth of position (i) and the paradox of supporting conclusions by unsound arguments. It does so by making compatibility a modal notion. Even then it has to admit supersound arguments for necessitives.

(iii) If "entails" is characterised by Anderson and Belnap's  $E$  (for clarity I will write " $\Rightarrow$ " for their entailment connective) and "if" is " $\Rightarrow$ " too, then the same modal fallacy is found as in (ii).

(iv) Lastly, let "entails" be represented by " $\Rightarrow$ " but "if" by " $\rightarrow$ ", the implication connective of Anderson and Belnap's calculi  $R$  and  $R^\Box$  of relevant implication.<sup>3</sup> The relation of compatibility which then arises,  $\sim(A \rightarrow \sim B)$ , is often written  $A \circ B$ , and " $\circ$ " is called "fusion". But to keep the concepts we are using absolutely clear, I will write " $x$ " for fusion, for " $\circ$ " was Lewis's symbol for his notion of consistency ( $A \circ B = \text{df } \sim(A \uparrow \sim B)$ ). In  $R^\Box$ ,  $AxB = \text{df } \sim(A \rightarrow \sim B)$ . (Note that  $AxB$  is equivalent to  $BxA$ .) Then at last, we shall see, we have the possibility of there being an invalid supersound argument with a false premise, whose conclusion it will follow must be true.

Clearly compatibility is not joint truth, as in (i) above. Because one expresses compatibility by such a phrase as "could both be true", some (e.g. Lewis) have supposed it a modal notion, leading to interpretation (ii). But such a phrase need not express modality; it can often import an inferential connection. For example, "It can't both be Tuesday and there be a logic lecture" does not mean that this combination of events is impossible, but simply that as things are, if it is Tuesday there is no logic lecture. The relation is one of implication, weaker than entailment.

<sup>3</sup> Too often (Meyer's thesis and *Entailment* § 8.20 are notable exceptions) the same symbol " $\rightarrow$ " is used for the implications of both  $E$  and  $R$ . A calculus adequate to represent both " $\Rightarrow$ " and " $\rightarrow$ " is Meyer's  $R^\Box$ , in which  $A \Rightarrow B$  is defined as  $\Box(A \rightarrow B)$  (though it was found to everyone's surprise to make " $\Rightarrow$ " differ from the entailment of  $E$ ).

There is therefore an alternative to the strengthening of "&" given in the shift in (ii) to "o"; we can move to "x". Position (iii) represents one of overkill, requiring both the inferential connection, and necessity. The present position, (iv), is, it seems, a sensible expression of validity and compatibility. But within it, the argument for the truth of the conclusion of every supersound argument commits no fallacy, nor does supersoundness entail soundness, and the paradox of supersoundness goes through.

From the standpoint of  $R^\square$ , the paradox of supersoundness is a special case of a more general phenomenon. For we can rehearse the puzzle of § 2 as follows: we wish to support a claim  $B$ , and to do so, we adduce another claim  $A$  from which  $B$  follows. To drive home the demonstration of  $B$ , we can present the reasoning thus:

$$(1) \frac{A \& A \Rightarrow B}{B}$$

Before becoming involved in any Lewis Carroll regresses, we now observe that to infer  $B$  from these premises we do not in fact need their joint truth. For their mere (non-modal) compatibility will do. That is, not only is (1) valid, so is

$$(2) \frac{A \times A \Rightarrow B}{B}$$

But the move from (1) to (2) is just a special case of a general feature of relevant logics, which we will note in § 7. Classically,  $P_1 \supset (P_2 \supset Q)$  is equivalent to  $(P_1 \& P_2) \supset Q$ . But relevantly we must distinguish

$$(3) P_1 \rightarrow (P_2 \rightarrow Q)$$

from

$$(4) (P_1 \& P_2) \rightarrow Q.$$

(4) does not entail (3). But if for some  $P_1, P_2$ , (3) holds, then we can use (3) rather than (4) to support  $Q$ . Suppose then that (3)  $P_1 \rightarrow (P_2 \rightarrow Q)$  is true. Then if  $P_1$  and  $P_2$  are each true, we can conclude that  $Q$  is true. But even if  $P_1$  or  $P_2$  is false, we can still properly infer  $Q$ , provided that

$P_1 \times P_2$  is true. That is, for the support of  $Q$  we can use the stronger result (3) – if it holds – rather than (4), and rely on the equivalence of (3) to  $(P_1 \times P_2) \rightarrow Q$  to support  $Q$  by two false premises which are simply compatible.

In the case of the soundness of an argument from  $A$  to  $B$ , (2) (corresponding to (3)) is available as an alternative to (1) (corresponding to (4)). So indeed the support of  $B$  requires only the supersoundness, and not necessarily the soundness, of the argument from  $A$ .

In the next section, I will explore whether there may not be a dialectical gain rather than paradox in the notion of supersound arguments understood in mode (iv).

### 5. Grasping the nettle

Logicians have concentrated on finding valid forms of argument. With such a form, one can support a conclusion by exhibiting an argument of this form with the right conclusion and showing that its premises are true. The conclusion is shown to follow validly from what is recognised to be true.

We now see that another technique would be to find forms of supersound argument. If there are any, then it will be possible to support a claim by showing it to be the conclusion of an instance of such a form. There will be no further need to show that its premises are true.

Among supersound arguments some will be only contingently supersound;  $(P \Rightarrow Q) \rightarrow \sim P$  will not be a theorem, but neither will its negation,  $(P \Rightarrow Q) \times P$ . With Hume we can recognise that belief in the contingent supersoundness of such an argument will often be outweighed by doubt as to the truth of its conclusion. But if we can find  $P, Q$  such that  $(P \Rightarrow Q) \times P$  is actually a theorem, then we will have a general form of supersound argument, one whose conclusion is true whatever the substitutions made for atoms in the particular formulae  $P$  and  $Q$ .

Now Simplification certainly fails for "x";  $A \times B$  does not imply  $A$  (or  $B$ ). But one might conjecture – indeed, Anderson did – that if  $A \times B$  is provable then so also are  $A$  and  $B$ .<sup>4</sup> If this were so, then every provably supersound argument would be sound, and our project would be doomed. The only unsound supersound arguments would be but contingently

<sup>4</sup> See *Entailment*, p. 333.

supersound and we could find no supersound forms which did not already have (logically) true premises, and were indeed valid too.

But Meyer showed Anderson's conjecture to be wrong. He exhibited a theorem  $A \times B$  of  $E$  such that  $B$  was not a theorem, although  $A$  was. The form of his example was

$$\vdash_E \sim(T \Rightarrow [ ] ((\sim T \Rightarrow T) \& \sim(T \Rightarrow (\sim T \Rightarrow T))))$$

where  $T$  is a theorem. Following the proof through, we can show that

$$\vdash_{R^{\square}} T \times S$$

where  $S$  is compounded from  $T$ , but is not provable.

The presence of  $T$  in Meyer's counterexample to Anderson's conjecture can be generalised. It can be shown that if  $A_1 \times A_2$  is provable then  $A_i$  is provable if, for  $i=1$  or  $2$ ,  $A_i$  is a necessitive (that is, for some  $B$ ,  $A_i$  is equivalent to  $\square B$ ). It follows that if  $(P \Rightarrow Q) \times P$  is provable then  $P \Rightarrow Q$  is provable. (I owe this observation to Meyer.) Hence all possibly unsound provably supersound argument forms are valid ones, whose premises are not necessitives. What we need to look for are theorems  $P \Rightarrow Q$  (of  $R^{\square}$ ) such that  $(P \Rightarrow Q) \times P$  is a theorem, but  $P$  is not. Any instance of such a form will decisively support  $Q$ , even though  $P$  may not be true. In § 8 I will use Meyer's counterexample  $T \times S$  to exhibit such an argument.

We must recognise however, that  $Q$ , entailed by the supersoundness of the argument whose conclusion it is, will, if the argument is formally supersound, be formally or logically true.  $Q$  will be a theorem (of  $R^{\square}$ ). The case is different with soundness. Suppose  $M$  is a theory whose logical basis is  $R^{\square}$ , e.g. a theory of arithmetic, or physics, or economics. Then although  $P \Rightarrow Q$  may be a theorem of  $R^{\square}$ , the truth of  $P$  may be no logical matter but result from the non-logical part of  $M$ . So  $Q$  will be a theorem of  $M$ , but not of  $R^{\square}$ . Study of the valid logical forms will allow us to proceed from one  $M$ -truth to another; whereas study of the supersound logical forms will tell us only about the logical truths, not the  $M$ -truths.

So on the one hand we have contingent supersoundness, open to Humean scepticism. On the other, formal supersoundness, of interest only to the logician. In between, there is the investigation of arguments which are necessarily supersound, but not logically or formally so. But of course, non-contingency is no mark of incorrigibility, and scepticism may

gain a foothold here too.

### 6. Material supersoundness

Those who are familiar with relevant logic will have recognised that the basis of the paradox of supersoundness is the theorem of  $R^{\square}$ ,  $(P \times (P \Rightarrow Q)) \Rightarrow Q$ . We have just found that the paradox is softened by the fact that whenever  $P \times (P \Rightarrow Q)$  is provable, so is  $P \Rightarrow Q$ . But  $R^{\square}$  also contains the theorem

$$(P \times (P \rightarrow Q)) \Rightarrow Q$$

and  $P \rightarrow Q$  is not a necessitive. It is therefore time to develop a variation on the notion of supersoundness.

We have required for an argument to be supersound that its validity be compatible with the truth of its premises. Validity is indeed a modal notion: an argument is valid when it is necessary that its premises imply its conclusion. Let us consider now a weaker relation, where validity is replaced by this (non-modal) implication relation. We say that an argument is *materially supersound* when the possibility of its premises simply implying (rather than entailing) its conclusion is compatible with the truth of those premises. As before, the classical account of the connectives will close off the paradox one way or the other; but a relevant account will result in the theorem that the conclusion of every materially supersound argument is true. That is, the truth of  $P \times (P \rightarrow Q)$  implies (indeed, entails) that of  $Q$  (and the provability – in  $R^{\square}$  – of  $P \times (P \rightarrow Q)$  implies the provability of  $Q$ ).

Here is an example of an unsound argument materially supersound in virtue of its form:

$$(I) \frac{p \rightarrow \sim p. \vee p}{(p \rightarrow \sim p. \vee p) \times (p \rightarrow \sim p. \vee p)}$$

In fact, (I) is valid. For generally  $\vdash_R A \rightarrow A \times A$ . (Suppose  $A$  were true but not  $A \times A$ , i.e.  $A \rightarrow \sim A$  were true. Then  $A$  and  $\sim A$  would both be true. So if  $A$  is true then  $A \times A$  is true, by *reductio*.)

To show that (I) is materially supersound, let us suppose that it is not, that is, that

$$(1) [(p \rightarrow \sim p. vp) \rightarrow (p \rightarrow \sim p. vp) \times (p \rightarrow \sim p. vp)] \rightarrow \sim(p \rightarrow \sim p. vp)$$

is true. Now by Identity,

$$A \times A \rightarrow A \times A$$

is always true, and therefore, by Exportation, so is

$$A \rightarrow (A \rightarrow A \times A),$$

in particular

$$(2) (p \rightarrow \sim p. vp) \rightarrow [(p \rightarrow \sim p. vp) \rightarrow (p \rightarrow \sim p. vp) \times (p \rightarrow \sim p. vp)].$$

From (1) and (2) by Hypothetical Syllogism, we can infer the truth of

$$(p \rightarrow \sim p. vp) \rightarrow \sim(p \rightarrow \sim p. vp)$$

whence follows that of

$$(3) (p \rightarrow \sim p. vp) \rightarrow [\sim(p \rightarrow \sim p) \ \& \ \sim p].$$

Now generally we have that  $A \vee B \rightarrow C \ \& \ D$  entails  $A \rightarrow C$  and  $B \rightarrow D$ . Hence we can infer from (3) the truth of

$$(p \rightarrow \sim p) \rightarrow \sim(p \rightarrow \sim p)$$

and

$$p \rightarrow \sim p$$

whence we can conclude that

$$p \rightarrow \sim p \ \text{and} \ \sim(p \rightarrow \sim p)$$

must both be true. But this is a contradiction. So by *reductio*, (1) must be false, that is,

$$[(p \rightarrow \sim p. vp) \rightarrow (p \rightarrow \sim p. vp) \times (p \rightarrow \sim p. vp)] \times (p \rightarrow \sim p. vp)$$

must be true. So (I) is materially supersound.

But (I) is not sound: its premise need not be true for suitable choice

of "p".<sup>5</sup> But since it is materially supersound, we can nonetheless infer that its conclusion is true for any choice of "p". In particular, it follows that, although  $(f \rightarrow t. vf)$  is not a theorem of  $R$  augmented by the Ackermann constants  $f$  and  $t$ ,  $(f \rightarrow t. vf) \times (f \rightarrow t. vf)$  is.<sup>6</sup>

But the point of this paper is not to reveal a new way of establishing conclusions which previously resisted demonstration, or even a new simpler means of proof.  $(p \rightarrow \sim p. vp) \times (p \rightarrow \sim p. vp)$  and its instance  $(f \rightarrow t. vf) \times (f \rightarrow t. vf)$  can be proved much more simply than by the above method. The problem I am posing is that the concepts available in relevant logic allow the possibility not only that true statements can be the conclusions of unsound arguments (which is too obvious for words), but that true statements can be shown to be true by the fact that they are the conclusions of certain types of unsound argument, namely supersound and materially supersound arguments.

### 7. Fusion

So the basis of this paper is the theorem of logics such as  $R$ ,

$$(P \times (P \rightarrow Q)) \rightarrow Q$$

and its counterpart in  $R^\square$  where " $\Rightarrow$ " replaces " $\rightarrow$ ". The paradox of §§ 2 and 6 is just another reason among many noted by others for asking: what is fusion, that is, "x"?

One possibility is that it is some sort of conjunction, since at least in  $R$  (though not in weaker systems, for example,  $R-W$ ), the truth of  $A \times B$  follows from that of  $A$  and  $B$ . But the converse inference, Simplification, fails, as I mentioned earlier. Adjunction and Simplification characterise extensional conjunction, "&". For "x", only what requires both  $A$  and  $B$  to establish follows from  $A \times B$ . What sort of conjunction is that?

<sup>5</sup>  $p \rightarrow \sim p. vp$  is a theorem of RM. It takes the value -0, which is not in the truth-filter, in the intensional monoid of *Entailment* § 28.2.4, when  $p$  is assigned the value -0.

<sup>6</sup> See R. Meyer, "Sentential Constants in  $R$ ", *Australian National University Research School of Social Sciences Logic Group Research Paper* No. 2 (1978), p. 39.  $f$  plays the role of -0 in the monoid of  $n.5$ ; and more generally is a greatest falsehood.

At least amongst relevant logicians, it is becoming common to distinguish two senses of disjunction. One is that in which "A or B" is equivalent to a conditional, and so supports Disjunctive Syllogism, but fails Addition; it is an inferential disjunction. The other sense of disjunction is truth-functional, and so supports Addition but fails Disjunctive Syllogism.<sup>7</sup> Lewis was (ironically) one of the first to make this distinction – but unfortunately his non-truth-functional disjunction was modal and irrelevant.

We need also to distinguish two conjunctions. One is extensional and truth-functional, and is related to the truth-functional disjunction ("v") as follows:

$$A \& B \leftrightarrow \sim(\sim A \vee \sim B)$$

and

$$(A \rightarrow C) \& (B \rightarrow C) \leftrightarrow ((A \vee B) \rightarrow C).$$

The other conjunction is fusion. It bears exactly the same relations to the intensional disjunction ("x"):

$$A \times B \leftrightarrow \sim(\sim A + \sim B)$$

and

$$(A \rightarrow C) \times (B \rightarrow C) \leftrightarrow ((A + B) \rightarrow C).$$

Moreover, we saw "x" is commutative, as is "&".

Two further observations will fully identify "x" for us. First: because Simplification fails for "x", "x" rejects Augmentation of Premises:  $A \times B \rightarrow C$  does not follow from  $A \rightarrow C$ . "x" expresses that conjunction of the premises of an argument where all are used. Now if one is looking for the valid forms of argument  $\Gamma \vdash A$  for which a deduction theorem is available, i.e.,  $\Gamma, B \vdash A$  iff  $\Gamma \vdash B \rightarrow A$ , then  $B$  must indeed in the appropriate sense to be found, have been used in the derivation of  $A$ ;  $B$  must be relevant to  $A$ . Augmentation of Premises introduces an irrelevancy.<sup>8</sup> Moreover, if

<sup>7</sup> See my "What is wrong with Disjunctive Syllogism?", *Analysis* 41 (1981), 66-70.

<sup>8</sup> See R. K. Meyer and M. A. McRobbie, "Firesets and Relevant Implication", *Australian National University Research School of Social Sciences Logic Group Research Paper No. 3* (1979), esp. p. 63 (forthcoming as "Multisets and Relevant Implication", *Australasian Journal of Philosophy*, July 1982).

simplicity is a criterion, one wishes when stating the valid forms to include only those premises in  $\Gamma$  which are indeed needed to obtain the conclusion  $A$ . The desirability of simplicity and the usefulness of a straightforward deduction theorem lead us therefore to demand that the premises be joined by "x", rather than by "&".

Secondly,  $P \rightarrow Q$  is true if and only if  $P \times \sim Q$  is not true; and so  $P \rightarrow Q$  is necessarily true if and only if  $P \times \sim Q$  cannot be true. Thus  $P$  entails  $Q$  iff  $P$  and  $\sim Q$  cannot both be true – provided that "and" there is read as "x". Fusion therefore expresses the connection between premises and conclusion required by the standard account of validity: an argument is valid if (and only if) its premises cannot be true and (fusion) its conclusion false. The test for validity is whether the truth of the premises is compatible with the falsity of the conclusion. Fusion is an inferential, and so intensional, conjunction.<sup>9</sup>

In particular, fusion is not extensional conjunction plus some further condition – e.g. that the conjuncts are relevantly connected. In the same way the correct account of validity is not strict implication plus some further condition – e.g. that the premises are relevant to the conclusion. " $\rightarrow$ " is based on "&"; change to "x" gives the correct account of validity.

Let us put all this together. We have an argument in which  $Q$  is inferred from  $P_1, \dots, P_n$ . The standard (classical, extensional) approach would require for the validity of this argument the necessary truth of

$$(P_1 \& \dots \& P_n) \supset Q,$$

i.e., of

$$\sim((P_1 \& \dots \& P_n) \& \sim Q).$$

The correct account is however gained by replacing all these extensional conjunctions by fusion, "x", to obtain

$$\sim((P_1 \times \dots \times P_n) \times \sim Q),$$

i.e.

<sup>9</sup> See my "Validity and the intensional sense of 'and'", *Australasian Journal of Philosophy*, 59 (1981), 301-7.

$$P_1 \rightarrow (P_2 \rightarrow (P_3 \rightarrow \dots (P_n \rightarrow Q)) \dots).$$

Fusion expresses both the link between the premises by which they are united in entailing the conclusion, and the link between them and the contradictory of the conclusion which must not hold if the argument is valid. The argument from  $P_1, \dots, P_n$  to  $Q$  is valid just when

$$(P_1 \times \dots \times P_n) \times \sim Q$$

cannot be true.

Consider the rule of Modus Ponens:  $A, A \rightarrow B \vdash B$ . This is usually expressed in axiomatic terms as  $(A \ \& \ (A \rightarrow B)) \rightarrow B$ . But we now appreciate that this binds the premises too weakly. Both should be used to obtain  $Q$  if this is the proper expression of a rule.<sup>10</sup> Rewriting the axiom as  $(A \times (A \rightarrow B)) \rightarrow B$ , it becomes equivalent to the law of Identity  $(A \rightarrow B) \rightarrow (A \rightarrow B)$ . This equivalence is a special case of the Deduction Theorem for Modus Ponens. So expressed, we see what is needed in terms of  $A$  and  $A \rightarrow B$  to obtain  $B$ : their compatibility, that is, the material supersoundness of the argument from  $A$  to  $B$ . The paradox of supersoundness is deeply embedded in  $R$ .

#### 8. An unsound supersound argument

I have yet to exhibit a possibly unsound provably supersound argument-form. Earlier we saw that a certain formula  $T \times S$  is a theorem of  $R^\square$ . Since  $S$  is compounded from  $T$ , and  $T$  is a theorem, this is not itself a theorem of the required form,  $(P \Rightarrow Q) \times P$ . But it can be adapted to yield one. Let  $T$  be  $(p \ \& \ q) \Rightarrow p$ , and consider

$$(II) \frac{\sim(\sim T \rightarrow T) \vee (T \rightarrow (\sim T \rightarrow T))}{(\sim T \rightarrow T) \vee \sim(T \rightarrow (\sim T \rightarrow T))}$$

Suppose (II) is not supersound, that is, that

<sup>10</sup> From the perspective of a weaker logic,  $R\text{-}W$ , the axiom  $(A \ \& \ (A \rightarrow B)) \rightarrow B$  is in fact false. In  $R\text{-}W$ , it is equivalent to  $W: A \rightarrow (A \rightarrow B) \rightarrow (A \rightarrow B)$ . It, and  $W$ , play an important role in the semantic paradoxes; see R. K. Meyer, R. Routley, and J. M. Dunn, "Curry's Paradox", *Analysis* 39 (1979), 124-8.

$$[(\sim(\sim T \rightarrow T) \vee (T \rightarrow (\sim T \rightarrow T))) \Rightarrow ((\sim T \rightarrow T) \vee \sim(T \rightarrow (\sim T \rightarrow T)))] \rightarrow \sim(\sim(\sim T \rightarrow T) \vee (T \rightarrow (\sim T \rightarrow T)))$$

is true. Now we can show in  $R^\square$  that

$$T \Rightarrow [(\sim(\sim T \rightarrow T) \vee (T \rightarrow (\sim T \rightarrow T))) \rightarrow ((\sim T \rightarrow T) \vee \sim(T \rightarrow (\sim T \rightarrow T)))]:$$

is true, by a  $\vee$ -elimination proof in which  $T$  is reiterated, since it is a necessitive, and is relevant to the derivation of the disjunctive consequent from each disjunct. Since  $T$  is a theorem, we can strengthen this to

$$T \Rightarrow [(\sim(\sim T \rightarrow T) \vee (T \rightarrow (\sim T \rightarrow T))) \Rightarrow ((\sim T \rightarrow T) \vee \sim(T \rightarrow (\sim T \rightarrow T)))]$$

So we have the truth of

$$T \rightarrow \sim(\sim(\sim T \rightarrow T) \vee (T \rightarrow (\sim T \rightarrow T)))$$

by transitivity, that is,

$$(1) T \rightarrow ((\sim T \rightarrow T) \ \& \ \sim(T \rightarrow (\sim T \rightarrow T))).$$

But Meyer shows in the course of his refutation of Anderson's conjecture (*Entailment*, pp. 333-4) how to prove, for any theorem  $T$ ,

$$T \Rightarrow \sim(T \Rightarrow ((\sim T \Rightarrow T) \ \& \ \sim(T \Rightarrow (\sim T \Rightarrow T)))).$$

Following his proof through, we can prove

$$T \rightarrow \sim(T \rightarrow ((\sim T \rightarrow T) \ \& \ \sim(T \rightarrow (\sim T \rightarrow T))))$$

and so

$$(2) \sim(T \rightarrow ((\sim T \rightarrow T) \ \& \ \sim(T \rightarrow (\sim T \rightarrow T))))$$

since  $T$  is a theorem. But (2) contradicts (1). So (II) must indeed be supersound.

But the premise of (II) can be false. Using Meyer's matrices (*ibid.*), the premise takes the undesigned value -0 when  $p$  takes the value +1 and  $q$  the value +2, while in these matrices  $R^\square$  is universally valid. So (II) is a possibly unsound provably supersound argument-form.

Moreover, the conclusion of (II) must be true, that is,



$$(\sim T \rightarrow T) \vee \sim(T \rightarrow (\sim T \rightarrow T)).$$

This is unsurprising when one reflects that  $\sim(\sim T \rightarrow T)$  entails  $\sim(T \rightarrow (\sim T \rightarrow T))$ , by *reductio*, since  $T$  is a theorem.

### 9. Conclusion

Have I given ammunition to the classicist – a stick with which to beat the relevantist? For the classicist may argue: if one divorces “if” from “entails”, and then takes “if” to be a proper restriction of “ $\supset$ ”, one will find oneself committed to the possibility of categorically supporting conclusions by invalid arguments with false premises – that is, to put it bluntly, by unsound arguments. One can avoid this paradox, he will say, by adhering to the classical view, and insisting that compatibility is a modal notion; for the only classical alternative would be to identify it with conjunction. Compatibility is then taken to be Lewis’ consistency operator, and the argument purporting to establish the truth of the conclusion of every supersound argument is rejected as committing a modal fallacy.

But the relevantist can reply that we have plenty of damning evidence against the classical paradigm, against, that is, identifying “if” with “ $\supset$ ”, and validity with “ $\supset$ ”. The existence of supersound arguments, and their role in the formal derivation of results, is, instead, an exciting breakthrough, a source of joy on releasing oneself from the classical shackles. When one says that an argument could be sound, that is, that its validity is compatible with the truth of its premises, one does not mean that the joint truth of its validity and the truth of its premises is logically possible, but that its invalidity does not follow from the truth of its premises, and the paradoxes of strict implication show that those notions are different. Often, we saw, when one says that  $P$  and  $\sim Q$  cannot both be true, one means not that the (extensional) conjunction of  $P$  and  $\sim Q$  is necessarily false, but that if  $P$  is true then so is  $Q$ . A possible formal system embodying such a non-classical conditional and compatibility-relation is  $R$ . So construed, there are formally supersound arguments, that is, supersound argument-forms, which can serve to demonstrate their conclusions, whether or not their premises are true. Moreover, although every such supersound argument-form is valid, there are also materially supersound argument-forms, which need not be valid. We considered an example of one. Though valid, it was not sound, and so the only way to use it to demon-

strate its conclusion, as indeed we did, was to rely on its proven material supersoundness.

A third view will reject  $R^\square$  as too strong in containing the theorem  $(P \times (P \Rightarrow Q)) \Rightarrow Q$  which forms the main part of this paper. One then retreats to  $E$  or to some weaker relative of  $R$  in which permutation fails quite generally for conditionals, thus rejecting the non-modal implicational and compatibility notions of  $R$ . For perhaps it is the combination of two tasks into one connective which does the damage in  $R$ . We there demand of one connective, “ $\times$ ”, that it satisfy both

$$A \times B \rightarrow C \text{ iff } A \rightarrow (B \rightarrow C)$$

and

$$A \times B \text{ iff } \sim(A \rightarrow \sim B).$$

In weaker calculi these tasks are performed by different connectives, and so again the demonstration of § 2 fails.

A similar set of three possibilities for reaction to the strict implication paradoxes faced Lewis after his admission of the theorems

$$\sim \diamond \sim Q \rightarrow (P \rightarrow Q)$$

and

$$\sim \diamond P \rightarrow (P \rightarrow Q)$$

into his calculi of strict implication: to stick to truth-functional logic, to stomach the “paradoxes”, and to look elsewhere for the correct account of entailment. Quine took the first, Lewis the second, and Ackermann the third. Among the three possibilities for reaction to the paradox of supersoundness I embrace the second position, and claim that the “paradox” of supersoundness in fact notes an intriguing fact about intensional conjunction.

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